Math 352 Real Analysis Exam 1

Instructions: Submit your work on any 5.

1. Let (M, d) be a metric space and $A \subset M$. Define $f: M \to \mathbb{R}$ by $f(x) = d(x, A)$. Show that f is Lipschitz. [20 pts]

2. Let (M, d) be a metric space and $A \subset M$. Show that $diam(\bar{A}) = diam(A)$.

[20 pts]

3. Prove that a normed vector space X is complete if and only if its closed unit ball $B = \{x \in X: \|x\| \leq 1\}$ is complete. [20 pts]

4. Prove that a uniformly continuous map sends Cauchy sequences into Cauchy sequences. [20 pts]

5. Suppose that $f: \mathbb{Q} \to \mathbb{R}$ is Lipschitz. Prove that f extends uniquely to a continuous function $g: \mathbb{R} \to \mathbb{R}$. [20 pts]

6. A function $f: \mathbb{R} \to \mathbb{R}$ is said to satisfy a Lipschitz condition of order $\alpha > 0$ if there is a constant $K < \infty$ such that $|f(x) - f(y)| \leq K|x - y|^\alpha$ for all x, y. Prove that such a function is uniformly continuous. [20 pts]

7. The Lipschitz condition is interesting only for $\alpha \leq 1$; show that a function satisfying a Lipschitz condition of order $\alpha > 1$ is constant. [20 pts]

8. Define $T: C[0,1] \to C[0,1]$ by

$$T(f)(x) = \int_0^x f(t)dt$$

Show that T is not a strict contraction while T^2 is. What is the fixed point of T? [20 pts]

9. Show that any open subset G of a metric space (M, d) is an F_σ set [20 pts]

10. Let $F \subseteq \mathbb{R}$ be a closed set. Show that $F = D(g)$ for some function $g: \mathbb{R} \to \mathbb{R}$. Recall that $D(g)$ is the set of discontinuities of g. [20 pts]

11. Let $\{K_n\}$ be a sequence of nested, connected sets in a metric space (M, d) such that $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$

(a) If we assume that the K_n are complete, does it follow that $K = \bigcap_{n=1}^{\infty} K_n$ is connected? Prove that this is the case or give a counter example.

(b) Assume that the K_n are compact. Prove that $K = \bigcap_{n=1}^{\infty} K_n$ is connected.

[20 pts]